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AUTHOR(S):

Kawabe, Jun

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A RIESZ TYPE REPRESENTATION THEOREM FOR RIESZ SPACE-VALUED POSITIVE LINEAR MAPPINGS

信州大学・工学部 河邊 淳 (JUN KAWABE)

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, SHINSHU UNIVERSITY, 4-17-1 WAKASATO, NAGANO 380-8553, JAPAN

ABSTRACT. Let X be a completely regular Hausdorff space and V a Dedekind complete Riesz space. The purpose of this note is to give a necessary and sufficient condition (tightness condition) which assures the validity of an analogue of the Riesz representation theorem for a positive linear mapping from $C(X)$ into V .

1. INTRODUCTION

Let X be a Hausdorff space and V a Dedekind complete Riesz space. Denote by $\mathcal{B}(X)$ the σ -field of all Borel subsets of X . A V -valued σ -measure on X is a finitely additive set function $\mu : \mathcal{B}(X) \rightarrow V$ such that $\mu(\bigcup_{n=1}^{\infty} A_n) = \sup_{n \in \mathbb{N}} \sum_{k=1}^n \mu(A_k)$ whenever $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{B}(X)$. If V possesses a Hausdorff vector topology τ for which each upper bounded monotone increasing sequence in V converges in the τ -topology to its least upper bound, V -valued σ -measures are ordinary topological vector space-valued measures that are fairly well understood; see Diestel and Uhl [2] and Kluvánek and Knowles [4]. But V need not possess any such topology; see Floyd [3].

The purpose of this note is to give a necessary and sufficient condition which assures that a given positive linear mapping T from $C(X)$, the space of all bounded, continuous, real-valued functions on X , into a Dedekind complete Riesz space V can be uniquely represented by a V -valued σ -measure μ on X such that $T(f) = \int_X f d\mu$ for all $f \in C(X)$. A successful analogue of the Riesz representation theorem was first proved by Wright [8, Theorem 4.1] and [10, Theorem 4.5] in the case that X is compact. See also [9, Theorem 1] for the case that X is locally compact. For the case that the representing measure μ is finitely additive, see Lipecki [5] and the literature therein. In Boccuto and Sambucini [1] a version of the above representation theorems has been discussed for "monotone integrals" with respect to Dedekind complete Riesz space-valued capacities.

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In Section 2 we recall some basic facts on Riesz spaces and give some preliminary results concerning regularities of Riesz space-valued σ -measures on a topological space. The results explained in the preceding paragraph are obtained in Section 3.

2. NOTATION AND PRELIMINARIES

All the topological spaces in this paper are Hausdorff and denote by \mathbb{R} and \mathbb{N} the set of all real numbers and the set of all natural numbers respectively.

2.1. Riesz spaces. A Riesz space is said to be *Dedekind complete* if every non-empty order bounded subset has a least upper bound. Every Dedekind complete Riesz space is Archimedean; see Schaefer [6, page 54].

Let V be a Riesz space and put $V^+ := \{u \in V : u \geq 0\}$. Given a net $\{u_\alpha\}_{\alpha \in \Gamma}$ in V we write $u_\alpha \downarrow u$ to mean that it is a decreasing net and $\inf_{\alpha \in \Gamma} u_\alpha = u$. The meaning of $u_\alpha \uparrow u$ is analogous.

Let $e \in V$ with $e > 0$. Denote by V_e the principal ideal generated by e , that is, $V_e := \{u \in V : |u| \leq re \text{ for some } r > 0\}$. Then, V_e is an AM-space with order unit e under the order unit norm $\|u\|_e := \inf\{r > 0 : |u| \leq re\}$, so that by the Kakutani-Krein theorem (see, for instance, [6, page 104]), it is isometrically and lattice isomorphic to $C(S)$, the space of all (bounded) continuous real-valued functions on a compact space S . Since V is Dedekind complete, so also is V_e . Hence S is Stonean, that is, the closure of every open subset of S is also open [6, page 108].

2.2. σ -measures. Let X be a topological space. Denote by $\mathcal{B}(X)$ the σ -field of all Borel subsets of X , that is, the σ -field generated by the open subsets of X . Denote by $C(X)$ the Banach lattice of all bounded, continuous, real-valued functions on X with supremum norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$ and by $B(X)$ the Banach lattice of all Borel measurable, bounded, real-valued functions on X with the same norm.

Let V be a Dedekind complete Riesz space. A finitely additive, positive set function $\mu : \mathcal{B}(X) \rightarrow V$ is called a σ -measure on X if it is σ -additive in the sense that whenever $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{B}(X)$ then $\mu(\bigcup_{n=1}^\infty A_n) = \sup_{n \in \mathbb{N}} \sum_{k=1}^n \mu(A_k)$. We emphasize that only measures taking positive values are considered in this paper.

As in the scalar case, every σ -measure has the monotone sequential continuity from above and from below, that is, whenever $\{A_n\}_{n \in \mathbb{N}}$ is an increasing (respectively a decreasing) sequence of sets in $\mathcal{B}(X)$ then $\mu(\bigcup_{n=1}^\infty A_n) = \sup_{n \in \mathbb{N}} \mu(A_n)$ (respectively $\mu(\bigcap_{n=1}^\infty A_n) = \inf_{n \in \mathbb{N}} \mu(A_n)$).

In Wright [8, 10] a V -valued integral with respect to a σ -measure μ is constructed and the successful analogues of the monotone convergence theorem and

the Lebesgue convergence theorem are obtained. We shall use the results there freely in this paper.

2.3. Regularities of σ -measures. As in usual measure theory on topological spaces we need to introduce some notions of regularities for Riesz space-valued σ -measures. Let X be a topological space and V a Dedekind complete Riesz space.

Definition 1. Let μ be a V -valued σ -measure on X .

- (i) μ is said to be *quasi-regular* if whenever G is an open subset of X then

$$\mu(G) = \sup \{ \mu(F) : F \subset G \text{ and } F \text{ is closed} \}.$$

- (ii) μ is said to be *quasi-Radon* if whenever G is an open subset of X then

$$\mu(G) = \sup \{ \mu(K) : K \subset G \text{ and } K \text{ is compact} \},$$

and it is said to be *tight* if the above condition holds for $G = X$.

- (iii) μ is said to be τ -smooth if whenever $\{G_\alpha\}_{\alpha \in \Gamma}$ is an increasing net of open subsets of X with $G = \bigcup_{\alpha \in \Gamma} G_\alpha$ then $\mu(G) = \sup_{\alpha \in \Gamma} \mu(G_\alpha)$.

Lemma 1. Let μ be a V -valued σ -measure on X .

- (i) μ is quasi-regular if and only if for each open subset G of X there exist a net $\{p_\alpha\}_{\alpha \in \Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{F_\alpha\}_{\alpha \in \Gamma}$ of closed subsets of X such that $F_\alpha \subset G$ and $\mu(G - F_\alpha) \leq p_\alpha$ for all $\alpha \in \Gamma$.
- (ii) μ is quasi-Radon if and only if for each open subset G of X there exist a net $\{p_\alpha\}_{\alpha \in \Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha \in \Gamma}$ of compact subsets of X such that $K_\alpha \subset G$ and $\mu(G - K_\alpha) \leq p_\alpha$ for all $\alpha \in \Gamma$.
- (iii) μ is tight if and only if there exist a net $\{p_\alpha\}_{\alpha \in \Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha \in \Gamma}$ of compact subsets of X such that $\mu(X - K_\alpha) \leq p_\alpha$ for all $\alpha \in \Gamma$.

Further, the above nets $\{F_\alpha\}_{\alpha \in \Gamma}$ and $\{K_\alpha\}_{\alpha \in \Gamma}$ can be chosen to be increasing.

Lemma 2. Let μ be a V -valued σ -measure on X . Then the following two conditions are equivalent:

- (i) μ is tight and quasi-regular.
- (ii) μ is quasi-Radon.

Lemma 3. Every quasi-Radon V -valued σ -measure μ on X is τ -smooth.

The following result can be proved as in the case of scalar measures; see for instance [7, Proposition I.3.2].

Proposition 1. Let μ be a τ -smooth V -valued σ -measure on X . Let $\{f_\alpha\}_{\alpha \in \Gamma}$ be a uniformly bounded, increasing net of lower semicontinuous real-valued functions on X . If $f = \sup_{\alpha \in \Gamma} f_\alpha$ is the pointwise supremum of f_α , then $\int_X f d\mu = \sup_{\alpha \in \Gamma} \int_X f_\alpha d\mu$.

Lemma 4. Assume that X is completely regular. Let μ and ν be τ -smooth V -valued σ -measures on X . If $\int_X f d\mu = \int_X f d\nu$ for each $f \in C(X)$ then $\mu = \nu$ on $\mathcal{B}(X)$.

3. AN ANALOGUE OF THE RIESZ REPRESENTATION THEOREM

Let X be a topological space and V a Dedekind complete Riesz space. In this section we give a necessary and sufficient condition (tightness condition) which assures the validity of an analogue of the Riesz representation theorem for a positive linear mapping from $C(X)$ into V .

First we extend Proposition 4.1 [8] to the case that X is not necessarily compact.

Proposition 2. Let X be a completely regular space and Y a compact space. Let $T : C(X) \rightarrow C(Y)$ be a positive linear mapping. Assume that there exist a net $\{p_\alpha\}_{\alpha \in \Gamma}$ in $C(Y)$ with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha \in \Gamma}$ of compact subsets of X such that $T(f) \leq p_\alpha$ whenever $\alpha \in \Gamma$ and $f \in C(X)$ with $0 \leq f \leq 1$ and $f(K_\alpha) = \{0\}$. Put $N := \{y \in Y : \inf_{\alpha \in \Gamma} p_\alpha(y) > 0\}$. Then there exists a mapping $\tilde{T} : B(X) \rightarrow B(Y)$ such that

- (i) \tilde{T} is positive and linear,
- (ii) for each $f \in C(X)$, $\tilde{T}(f)(y) = T(f)(y)$ for all $y \notin N$,
- (iii) if $\{f_n\}_{n \in \mathbb{N}}$ is a uniformly bounded sequence in $B(X)$ which converges pointwise to f , then $f \in B(X)$ and

$$\tilde{T}(f)(y) = \lim_{n \rightarrow \infty} \tilde{T}(f_n)(y) \text{ for all } y \in Y,$$

- (iv) if f is a lower semicontinuous real-valued function on X , then

$$\tilde{T}(f)(y) = \sup\{T(g)(y) : 0 \leq g \leq f, g \in C(X)\} \text{ for all } y \notin N,$$

and hence $\tilde{T}(f)$ is lower semicontinuous on $Y - N$.

From Proposition 2 we naturally reach the following definition.

Definition 2. Let X be a topological space and V a Riesz space. We say that a positive linear mapping $T : C(X) \rightarrow V$ satisfies the *tightness condition* if there exist a net $\{p_\alpha\}_{\alpha \in \Gamma}$ in V with $p_\alpha \downarrow 0$ and a net $\{K_\alpha\}_{\alpha \in \Gamma}$ of compact subsets of X such that $T(f) \leq p_\alpha$ whenever $\alpha \in \Gamma$ and $f \in C(X)$ with $0 \leq f \leq 1$ and $f(K_\alpha) = \{0\}$.

Let S be a compact Stonean space. Denote by \mathcal{M} the σ -ideal of all meager Borel subsets of S . Let κ be a canonical $C(S)$ -valued σ -measure on S such that

- ($\kappa 1$) \mathcal{M} is the kernel of κ ,
- ($\kappa 2$) $\kappa(E) = \chi_E$ for all clopen subset E of S .

The existence of κ follows from [8, page 118] and κ is called the *Birkhoff-Ulam $C(S)$ -valued σ -measure* on S .

The following lemma has been already given in [8] implicitly.

Lemma 5. *Let κ be the Birkhoff-Ulam $C(S)$ -valued σ -measure on S . Then $\int_S f d\kappa = f$ for all $f \in C(S)$.*

We are now ready to give an analogue of the Riesz representation theorem for a Dedekind complete Riesz space-valued positive linear mapping.

Theorem 1. *Let X be a completely regular space and V a Dedekind complete Riesz space. Let $T : C(X) \rightarrow V$ be a positive linear mapping. Then the following two conditions are equivalent:*

(i) *T satisfies the tightness condition.*

(ii) *There exists a quasi-Radon V -valued σ -measure μ on X such that*

$$(1) \quad T(f) = \int_X f d\mu \quad \text{for all } f \in C(X).$$

Further, the μ is determined by (1) and the quasi-Radonness of μ .

The tightness condition in the above theorem is automatically satisfied if X is compact, and hence Theorem 1 reduces to a special case of the results obtained in [8, Theorem 4.1] and [10, Theorem 4.5]. See also [9, Theorem 1]. However, our work will be needed to develop the theory of the weak order convergence of Riesz space-valued σ -measures, in which we usually assume that the involved σ -measures are defined on metric spaces or more generally on completely regular spaces. As an application in this light, we shall show in a later work that the operation making the Borel product of two Riesz space-valued σ -measures is jointly continuous with respect to the weak order convergence of σ -measures.

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DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, SHINSHU UNIVERSITY, 4-17-1
WAKASATO, NAGANO 380-8553, JAPAN

E-mail address: jkawabe@gipwc.shinshu-u.ac.jp